Chapter One: Analysis of doubly reinforced rectangular beam

3rd stage

Analysis of rectangular beams with tension and compression reinforcements (Doubly reinforced beam)

The steel that is occasionally used on the compression sides of beams is called compression steel, and beams with both tensile and compressive steel are referred to as **doubly reinforced beams**.

When the beam cross section is limited because of architectural or other considerations. It may happen that the concrete cannot develop the compression force required to resist the given bending moment; in this case, reinforcement is added in the compression zone, resulting in doubly reinforcement beams.

As (Reinforcement area for tension)

\[ \rho = \frac{A_s}{bd} \]

As` (Reinforcement area for compression)

\[ \rho' = \frac{A_{s'}}{bd} \]

- There are four main reasons for using compression reinforcement in beams:
  1. **Reduce sustained–load deflection (Long Term Deflection)**
     
     It has been found that the inclusion of some compression steel will reduce the long-term deflections of members.
It is useful to note, there are two types of deflections:

**a. Immediate (short) deflection**

This deflection occurs immediately upon the application of a load.

**b. Long term deflection**

Take into account the shrinkage and creep movements.

- Calculation of deflection of beam and compare it with allowable limits in ACI Code is under serviceability requirements of beam and will be studying briefly in fourth year (senior course).

2. **Stirrups Supports**

Continues compression bars are also helpful for positioning stirrups (by tying them to the compression bars) and keeping them in place during concrete placement and vibration.

Minimum rebars in compression zone to support stirrups
3. Increase ductility

Compression reinforcement increases not only the resisting moments of concrete sections but also the amount of curvature. This means that the ductility of such sections will be appreciably increased. Though expensive, compression steel makes beam tough and ductile, enabling them to withstand large moment’s and stress reversals such as might occur during earthquakes.

Ductile concrete beam (favorite)

Brittle Concrete beam (not favorite)

Stress-strain diagram for brittle and ductile materials.
4. Changing the failure mode from compression to tension failure

According to ACI Code, all beams are to be designed for yielding of the tension steel, and thus $\rho \leq \rho_{\text{max}}$.

As was mentioned before sometimes beam cross section is limited because of architectural or other considerations. So addition steel in compression zone is required to be added to change the state of failure from compression to tension.

It is useful to remember there are two types of failure in concrete beam; the first one is the failure of tension zone of concrete and yielding of steel before compression zone and this type of failure is required and permitted in ACI code, consequently this mode of failure will give early notice for beam before failure which in turn give times for people to leave the building before collapse, however the second type of failure do not give any warning before failure and this happened when the compression zone fails before tension zone and this type is not permitted in ACI code.
Chapter One: Analysis of doubly reinforced rectangular beam

3rd stage

Tension failure of beam (permitted in ACI Code)

Compression failure of beam (not permitted in ACI Code)
1. Check the reason for using of compression reinforcement

   find  \( \rho = \frac{A_s}{bd} \)

   and  \( \rho_{max} = 0.85\beta_1 \frac{f'c'}{f_y} \frac{\epsilon_u}{\epsilon_u + 0.004} \)

   where  \( \epsilon_u = 0.003 \)

   If  \( \rho \leq \rho_{max} \)

   Then the compression reinforcement has been used either to reduce sustained-load deflection or to stirrups support or to increase ductility and it is effect can be neglected in the beam design.

   Then analysis the beam as singly reinforcement □

   Else, if  \( \rho > \rho_{max} \)

   Then the compression reinforcement has been used to change the mode of failure from compression to tension failure, and then this reinforcement must be included in the beam analysis then go to step 2.

2. Calculate  \( \rho_{\sim max} \)

   \( \rho_{\sim max} = \rho_{max} + \rho \frac{f_s}{f_y} \)

   \( \rho_{\sim} = \frac{A_s}{bd} \)

   Where  \( f_s \) is stress in the compression reinforcement. It can be computed from relation below:

   \( f_s = E_s \left[ \left( \frac{\epsilon_u}{d} \right) (\epsilon_u + 0.004) \right] \leq f_y \)  where  \( E_s = 200,000 \) Mpa and  \( \epsilon_u = 0.003 \)

   If  \( \rho \leq \rho_{\sim max} \) O.k

   If  \( \rho > \rho_{\sim max} \) section is not O.k

3. Calculate  \( \rho_{cy} \)

   \( \rho_{cy} = 0.85\beta_1 \frac{f'c'}{f_y} \frac{d}{d} \frac{\epsilon_u}{\epsilon_u - \epsilon_y} + \rho \)

   If  \( \rho_{cy} \leq \rho \) Then calculate the moment according to step 4.1
Chapter One: Analysis of doubly reinforced rectangular beam

3rd stage

Else if
ρ_{cy} > ρ go to step 4.2

4.1 Compute section nominal moment M_n when (ρ_{cy} ≤ ρ)

\[ M_n = M_{n1} + M_{n2} = A_s f_y (d-d') + (A_s - A_s') f_y (d-\frac{a}{2}) \]

\[ a = \frac{(A_s - A_s') f_y}{0.85 f_{c'} b} \]

Calculate \( \varnothing \)

\[ c = \frac{a}{\beta_1} \]

\[ \epsilon_t = \frac{d t - c}{c} \epsilon_u \]

where: \( \epsilon_u = 0.003 \)

- If \( \epsilon_t \geq 0.005 \), then \( \varnothing = 0.9 \)
- If \( \epsilon_t < 0.005 \)

\[ \varnothing = 0.483 + 83.3 \epsilon_t \]

Calculate \( \varnothing M_n \) ■

4.2 Compute section nominal moment M_n when (ρ_{cy} > ρ)

\[ M_n = M_{n1} + M_{n2} = 0.85 f_{c'} ab (d-d') + A_s' f_s' (d-d') \]

\( f_s' < f_y \)

Calculate \( f_s' \):

\[ f_s' = \epsilon_u \times E_s \times \left( c - \frac{d'}{c} \right) \]

Where c:

\[ c = \sqrt{Q + R^2} - R \]

\[ Q = \frac{600 d' A_s}{0.85 \beta_1 f_{c'} b} \]

And

\[ R = \frac{600 A_s' - f_y A_s}{1.7 \beta_1 f_{c'} b} \]

\[ a = \beta_1 c \]

Calculate \( \varnothing \)

\[ \epsilon_t = \frac{d t - c}{c} \epsilon_u \]

where: \( \epsilon_u = 0.003 \)

- If \( \epsilon_t \geq 0.005 \), then \( \varnothing = 0.9 \)
- If \( \epsilon_t < 0.005 \)

\[ \varnothing = 0.483 + 83.3 \epsilon_t \]

Calculate \( \varnothing M_n \) ■
Example 1: Check the adequacy of beam shown in figure below and compute its design strength according to ACI Code. Use $f_c = 20$ MPa and $f_y = 300$ MPa
A bar of 25mm = 490mm$^2$

Solution:

1. Check the reason for using of compression reinforcement

   \[
   \rho = \frac{A_s}{b d} = \frac{4 \times 490}{300 \times 450} = \frac{1960}{300 \times 450} = 14.54 \times 10^{-3} 
   \]

   and

   \[
   \rho_{\text{max}} = 0.85 \beta_1 \frac{f_c}{f_y} \frac{\epsilon_u}{\epsilon_u + 0.004} 
   \]

   where $\epsilon_u = 0.003$

   \[
   \rho_{\text{max}} = 0.85 \times 0.85 \times \frac{20}{300} \times \frac{0.003}{0.003 + 0.004} = 20.6 \times 10^{-3}
   \]

   \[\rho < \rho_{\text{max}}\]

   Then the compression reinforcement has been used either to reduce sustained-load deflection or to stirrups support or to increase ductility and it is effect can be neglected in the beam design.

   Then the beam can be analysis as singly reinforcement.
Chapter One: Analysis of doubly reinforced rectangular beam

2. Calculate $\varnothing$

\[
a = \frac{A_s f_y}{0.85 f'_c b} = \frac{1960 \times 300}{0.85 \times 20 \times 300} = 115 \text{ mm}
\]
\[
c = \frac{a}{\beta_1} = \frac{115}{0.85} = 135 \text{ mm}
\]
\[
\varepsilon_t = \frac{d - c}{c} \times \varepsilon_u \quad \text{where: } \varepsilon_u = 0.003
\]
\[
\varepsilon_t = \frac{450 - 135}{135} \times 0.003 = 7 \times 10^{-3} > 0.005 \quad \therefore \quad \varnothing = 0.9
\]

3. Calculate $\varnothing M_n$

\[
\varnothing M_n = \varnothing A_s f_y (d - \frac{a}{2}) = 0.9 \times 1960 \times 300 \times (450 - \frac{115}{2}) \times 10^{-6} = 207 \text{ kN.m}
\]

**Example 2**: Check the adequacy of beam shown in figure below and compute its design strength according to ACI Code. Use $f'_c = 20 \text{ MPa}$ and $f_y = 300 \text{ Mpa}$

**Solution**:

1. Check the reason for using of compression reinforcement

\[
\rho = \frac{A_s}{b d} = \frac{6 \times 490}{250 \times 450} = \frac{2940}{250 \times 450} = 26.1 \times 10^{-3}
\]

\[
\rho_{\max} = 0.85 \beta_1 \frac{f_c}{f_y} \frac{e_u}{e_u + 0.004} \quad \text{where } \varepsilon_u = 0.003
\]

\[
\rho_{\max} = 0.85 \times 0.85 \times \frac{20}{300 \times 0.003 + 0.004} = 20.6 \times 10^{-3}
\]
\( \rho > \rho_{\text{max}} \)

Then the compression reinforcement has been used to change the mode of failure from compression to tension failure, and then this reinforcement must be included in the beam analysis.

2. Calculate \( \rho'_{\text{max}} \)

\[
\rho'_{\text{max}} = \rho_{\text{max}} + \rho \frac{f_s}{f_y}
\]

\[
\rho' = \frac{As}{bd} = \frac{3 \times 490}{250 \times 450} = 13.1 \times 10^{-3}
\]

\[
f'_s = E_s \left[ \varepsilon_u - \frac{d}{d'} (\varepsilon_u + 0.004) \right] \leq f_y \text{ where } E_s=200,000 \text{ Mpa and } \varepsilon_u=0.003
\]

\[
f'_s = 200,000 \left[ 0.003 - \frac{50}{450} (0.003 + 0.004) \right] = 444 > f_y
\]

\( \therefore f_s = f_y = 300 \text{ MPa} \)

\[
\rho'_{\text{max}} = \rho_{\text{max}} + \rho \frac{f_s}{f_y} = 20.6 \times 10^{-3} + 13.1 \times 10^{-3} \times \frac{300}{300} = 33.7 \times 10^{-3}
\]

\( \rho \leq \rho'_{\text{max}} \text{ O.k} \)

3. Calculate \( \rho_{cy} \)

\[
\rho_{cy} = 0.85\beta_1 \frac{f_c}{f_y} \times \frac{d}{d'} \frac{\varepsilon_u}{\varepsilon_u - \varepsilon_y} + \rho'
\]

\[
\rho_{cy} = 0.85 \times 0.85 \times \frac{20}{300} \times \frac{50}{450} \times \frac{0.003}{300} - \frac{300}{200,000} + 13.1 \times 10^{-3} = 23.8 \times 10^{-3} < \rho
\]

\( \therefore f_s = f_y = 300 \text{ MPa} \)

4. Compute section nominal moment \( M_n \) when \( \rho_{cy} \leq \rho \)

\[
M_n = M_{n1} + M_{n2} = A_S f_y (d-d') + (A_S-A_S') f_y (d-\frac{a}{2})
\]

\[
a = \frac{(A_S-A_S') f_y + (A_S-A_S') f_y (d-\frac{a}{2})}{0.85 f_c' + b} = \frac{(2940-1470) \times 300}{0.85 \times 20 \times 250} = 104 \text{ mm}
\]

\[
M_n = 1470 \times 300 (450-50) + (2940-1470) \times 300 (450-\frac{104}{2})
\]

\[\text{Mn}=176.4 \times 10^6 \text{ kN.m} + 175.5 \times 10^6 \text{ kN.m} = 352 \text{ kN.m} \]

Calculate \( \phi \)

\[
c = \frac{a}{\beta_1} = \frac{104}{0.85} = 122 \text{ mm}
\]

\[
\varepsilon_t = \frac{d_t-c}{c} \varepsilon_u = \frac{475-122}{122} \times 0.003 = 8.68 \times 10^{-3} > 0.005 \quad \therefore \phi = 0.9
\]

Calculate \( \phi M_n = 0.9 \times 352 = 317 \text{ kN.m} \)
**Example 3:** recheck the adequacy of example 2 above but with $d` = 65$ mm,

**Solution:**

1. Check the reason for using of compression reinforcement

   \[ \rho = \frac{A_s}{bd} = \frac{6 \times 490}{250 \times 450} = \frac{2940}{250 \times 450} = 26.1 \times 10^{-3} \]

   and \[ \rho_{max} = 0.85 \beta \frac{f_c}{f_y} \frac{\epsilon_u}{\epsilon_u + 0.004} \]

   where $\epsilon_u = 0.003$

   \[ \rho_{max} = 0.85 \times 0.85 \times \frac{20}{300} \frac{0.003}{0.003 + 0.004} = 20.6 \times 10^{-3} \]

   \[ \rho > \rho_{max} \]

   Then the compression reinforcement has been used to change the mode of failure from compression to tension failure, and then this reinforcement must be included in the beam analysis.

2. Calculate $\rho_{\max}$

   \[ \rho'_{\max} = \rho_{\max} + \rho_s \frac{f_s}{f_y} \]

   \[ \rho' = \frac{A_s}{bd} = \frac{3 \times 490}{250 \times 450} = 13.1 \times 10^{-3} \]

   \[ f_s' = E_s \left[ \epsilon_u - \frac{d'}{d} (\epsilon_u + 0.004) \right] \leq f_y \quad \text{where } E_s = 200,000 \text{ Mpa and } \epsilon_u = 0.003 \]

   \[ f_s' = 200,000 \left[ 0.003 - \frac{65}{450} (0.003 + 0.004) \right] = 398 > f_y \]

   \[ \therefore f_s' = f_y = 300 \text{ MPa} \]

   \[ \rho'_{\max} = \rho_{\max} + \rho_s \frac{f_s}{f_y} = 20.6 \times 10^{-3} + 13.1 \times 10^{-3} \times \frac{200}{300} = 33.7 \times 10^{-3} \]

   \[ \rho \leq \rho'_{\max} \text{ O.k} \]

3. Calculate $\rho_{cy}$

   \[ \rho_{cy} = 0.85 \beta \frac{f_c}{f_y} \frac{d'}{f_y} \frac{\epsilon_u}{d \epsilon_u - \epsilon_y} + \rho' \]

   \[ \rho_{cy} = 0.85 \times 0.85 \times \frac{20}{300} \times \frac{65}{450} \frac{0.003}{\frac{300}{200,000}} + 13.1 \times 10^{-3} \]

   \[ \rho_{cy} = 13.9 \times 10^{-3} + 13.1 \times 10^{-3} > 27 \times 10^{-3} > \rho \]

   \[ \therefore f_s' < f_y \]

4. Compute section nominal moment $M_n$ when ($\rho_{cy} > \rho$)

   \[ M_n = M_{n1} + M_{n2} = 0.85 f_c' ab \left( d - \frac{a}{2} \right) + A_s f_s' (d - d') \]

   Calculate $f_s'$:
\[ f_s' = \varepsilon_u * E_s \left( \frac{c - d'}{c} \right) \]

Where \( c \):
\[ c = \sqrt{Q + R^2} - R \]
\[ Q = \frac{600d'As'}{0.85\beta_1f_c'b} = \frac{600*65*1470}{0.85*0.85*20*250} = 15870 \]

And
\[ R = \frac{600As' - fyAs}{1.7\beta_1f_c'b} = \frac{600*1470 - 300*2940}{1.7*0.85*20*250} = 0 \]
\[ c = \sqrt{Q + R^2} - R = \sqrt{15870 + 0^2} - 0 = 126 \text{ mm} \]

\[ f_s' = 0.003 * 200,000 * \left( \frac{126 - 65}{126} \right) = 290 \text{ MPa} < f_y \text{ O.k} \]
\[ a = \beta_1c = 0.85*126 = 107 \text{ mm} \]

\[ M_n = M_{n1} + M_{n2} = 0.85f_c'b(d - \frac{a}{2}) + A_s f_s' (d - d') \]

\[ M_n = 0.85*20*107*250 (450 - \frac{107}{2}) + 1470 * 290 * (450 - 65) \]

\[ M_n = 180.3 * 10^6 \text{ N.m} + 164.1 * 10^6 \text{ N.mm} = 344 * 10^6 \text{ N.mm} = 344 \text{ kN.m} \]

Calculate \( \phi \)
\[ \varepsilon_t = \frac{dt - c}{c} \varepsilon_u \quad \text{where: } \varepsilon_u = 0.003 \]
\[ \varepsilon_t = \frac{475 - 126}{126} * 0.003 = 8.3 * 10^{-3} > 0.005 \quad \therefore \phi = 0.9 \]

Calculate \( \phi \) \( M_n \)
\[ \phi \ M_n = 0.9 * 344 = 310 \text{ kN.m} \]
Example 4: Check the adequacy of the beam shown below and compute its design strength according to ACI Code. Assume that:

1. $f_{c'} = 34.5$ MPa
2. $f_y = 414$ MPa
3. Area of bar No.25mm = 510 mm$^2$
4. Area of bar No.32mm = 819 mm$^2$

Solution:

1. Check the reason for using of compression reinforcement

   \[
   \rho = \frac{A_s}{bd} = \frac{8 \times 819}{356 \times 660} = \frac{6552}{356+660} = 27.9 \times 10^{-3}
   \]

   \[
   \rho_{\text{max}} = 0.85\beta_1 \frac{f_{c'}}{f_y} \frac{\varepsilon_u}{\varepsilon_u + 0.004}
   \]
   where $\varepsilon_u = 0.003$

   \[
   \rho_{\text{max}} = 0.85 \times 0.804 \times \frac{34.5}{414} \times \frac{0.003}{0.003 + 0.004} = 24.4 \times 10^{-3}
   \]

   $\rho > \rho_{\text{max}}$

   Then the compression reinforcement has been used to change the mode of failure from compression to tension failure, and then this reinforcement must be included in the beam analysis.

2. Calculate $\rho_{\text{max}}'$

   \[
   \rho_{\text{max}}' = \rho_{\text{max}} + \rho \frac{f_s}{f_y}
   \]
\[ \rho = \frac{2 \times 510}{356 \times 660} = \frac{1020}{356 \times 660} = 4.34 \times 10^{-3} \]

\[ f_s = E_s \left[ \varepsilon_u - \frac{d}{d} (\varepsilon_u + 0.004) \right] \leq f_y \quad \text{where} \quad E_s = 200,000 \text{ Mpa and } \varepsilon_u = 0.003 \]

\[ f_s = 200,000 \left[ 0.003 - \frac{76}{660} (0.003 + 0.004) \right] = 438.78 \text{ MPa} > 414 \text{ MPa} \]

\[ \therefore f_s = f_y = 414 \text{ MPa} \]

\[ \rho_{\text{max}} = \rho_{\text{max}} + \rho_{\text{fs}} \]

\[ \rho_{\text{max}} = 24.4 \times 10^{-3} + 4.34 \times 10^{-3} = 28.9 \times 10^{-3} \]

\[ \rho < \rho_{\text{max}} \quad \text{O.k} \]

3. Calculate \( \rho_{cy} \)

\[ \rho_{cy} = 0.85 \beta_1 f_c' \frac{d}{d} \varepsilon_u + \rho \]

\[ \rho_{cy} = 0.85 \times 0.804 \times 34.5 \times 660 \times 0.003 + \frac{414}{200,000} \times 4.34 \times 10^{-3} = 25.5 \times 10^{-3} \]

\[ \rho_{cy} \leq \rho \]

\[ \therefore f_s = f_y = 414 \text{ MPa} \]

4. Compute section nominal moment \( M_n \) when \( \rho_{cy} \leq \rho \)

\[ M_n = M_{n1} + M_{n2} = A_s f_y (d - d') + (A - A_s) f_y (d - \frac{a}{2}) \]

\[ a = \frac{(A - A_s) f_y}{0.85 c'} = \frac{(6552 - 1020) \times 414}{0.85 \times 34.5 \times 356} = 219 \text{ mm} \]

\[ M_n = 1020 \times 414 \times (660 - 76) + (6552 - 1020) \times 414 \times \frac{219}{2} \]

\[ M_n = 247 \times 10^6 \text{ N.mm} + 1261 \times 10^6 \text{ N.mm} = 1508 \text{ kN.m} \]

Calculate \( \varnothing \)

\[ c = \frac{a}{\beta_1} = \frac{219}{0.804} = 272 \text{ mm} \]

\[ \epsilon_t = \frac{dt - c}{c} \varepsilon_u \]

where: \( \varepsilon_u = 0.003 \)

\[ \epsilon_t = \frac{685 - 272}{272} \times 0.003 = 4.55 \times 10^{-3} < 0.005 \]

\[ \varnothing = 0.483 + 83.3 \times \epsilon_t = 0.86 \]

Calculate \( \varnothing M_n \)

\[ \varnothing M_n = 0.86 \times 1508 = 1297 \text{ kN.m} \]